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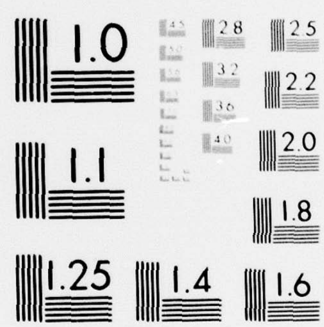

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NONLINEAR OSCILLATIONS AND BOUNDARY-VALUE PROBLEMS FOR HAMILTONIAN SYSTEMS

Frank H. Clarke* and I. Ekeland**

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ABSTRACT

We prove existence of solutions to various boundary-value problems for nonautonomous Hamiltonian systems with forcing terms:

$$-\dot{p}(t) = H'_x(t, x(t), p(t)) + f(t)$$

$$\dot{x}(t) = H'_p(t, x(t), p(t)) + g(t) \quad .$$

Among these problems is that of <u>periodic</u> solutions: $x(t + T) = x(t)$, $p(t + T) = p(t)$. A special study is made of the classical case in which $H(x,p)$ has the form $V(x) + |p|^2/2$, potential plus kinetic energy, where the existence of an infinite family of free harmonics is proven. The approach throughout is via a variational principle involving a new, dual action integral.

AMS(MOS) Subject Classification: 34C15, 34C25

Key Words: Nonlinear vibrations, periodic trajectories, variational principle, action integral, Hamiltonian system

Work Unit Number 1 - Applied Analysis

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SIGNIFICANCE AND EXPLANATION

Hamilton's differential equations are basic in the study of theoretical mechanics. A particular class of motions of interest for such systems of equations are the periodic ones, which correspond to oscillations (vibrations) of the underlying physical system; the absence of such motions is usually associated with resonance phenomena. In this paper we give conditions on the Hamiltonian function H which guarantee the existence of periodic orbits, as well as other more general types of motions. One distinction with previous work on the subject is that we consider forced vibrations arising from external driving forces; another is that the solutions in question are characterized directly as the solutions of a specific minimization problem (i.e., we obtain a "variational principle"), a feature which could prove useful for computational purposes.

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NONLINEAR OSCILLATIONS AND BOUNDARY-VALUE PROBLEMS

FOR HAMILTONIAN SYSTEMS

Frank H. Clarke* and I. Ekeland**

§1. Introduction: a dual action principle.

This article is devoted to the study of certain boundary-value problems for Hamiltonian systems of the form

$$(1.1)\qquad \begin{cases} -\dot{p}(t) = H_x'(t, x(t), p(t)) + f(t) \\ \dot{x}(t) = H_p'(t, x(t), p(t)) + g(t) \end{cases}$$

Here, we interpret f and g as exterior forcing terms, while the (relatively weak) dependence of H on t can be viewed as due to the presence of time-varying parameters in the system itself. Of particular interest among the boundary-value problems associated with (1.1) is the question of T-periodic solutions: $x(t + T) = x(t)$, $p(t + T) = p(t)$. Our results, which are of a more general nature, prove the existence on certain intervals $[0,T]$ of solutions (x,p) to (1.1) satisfying various boundary conditions.

The Hamiltonians $H(t,x,p)$ that we consider are convex in (x,p) for each t, and in these same variables exhibit growth that is superlinear but no more than quadratic (of course, we are stating assumptions and facts loosely at this point). The superquadratic case, which can be treated by modifying our approach somewhat along the lines of [7], will appear elsewhere. Our approach is a refinement of the one employed in [4] and later in [6], and involves a new "dual action principle." To describe this principle, let us recall the conjugate convex function G corresponding to H:

$$(1.2)\qquad G(t,y,q) = \sup\{y \cdot x + q \cdot p - H(t,x,p) : (x,p) \in \mathbb{R}^n \times \mathbb{R}^n\} .$$

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Consider now the following variational problem: to minimize

$$\int_0^T \{-p \cdot \dot{x} + G(t, -\dot{p} - f, \dot{x} - g)\}\, dt \tag{1.3}$$

over functions (x,p) satisfying some given set of boundary conditions at 0 and T. The dual action principle alluded to is the following: if $(\tilde{x},\tilde{p})$ is an extremal for the functional (1.2), then there exist translates (x,p) of $(\tilde{x},\tilde{p})$ which satisfy the Hamiltonian system (1.1) (i.e. for certain constants σ_0, s_0, the functions $x = \tilde{x} + \sigma_0$, $p = \tilde{p} + s_0$ satisfy (1.1)). This is Lemma 2.1 in §2, where however the change of variables

$$\dot{y} = \dot{x} - g \quad , \quad \dot{q} = \dot{p} + f$$

has been applied to the functional.

The function G defined by (1.2) may not be differentiable, and the reader may wonder how to interpret "extremal" in the above. As we shall see, the approach used here requires no differentiability; in fact, we actually consider a more general "Hamiltonian inclusion" of which (1.1) is a special case (see 2.1). The main advantage of the dual action principle in our present setting is that the functional (1.3) will admit a minimum, in sharp contrast to the usual action integral

$$\int_0^T \{-p \cdot \dot{x} + H(t,x,p)\}\, dt$$

which is bounded neither above nor below. In consequence, the trajectories of (1.1) are not only known to exist, but are furthermore characterized as solutions to a specific minimization problem, a fact that could prove useful for computation.

In §2 we prove the existence, for all T suitably bounded above, of solutions (x,p) of (1.1) for which $x(T) - x(0)$, $p(T) - p(0)$ are arbitrarily prescribed. (Thus, as a special case, we obtain periodicity). A further result yields existence of solutions for which $x(0)$ and $x(T)$ assume arbitrarily prescribed values.

In §3, we investigate the case when there is no forcing term on the right-hand side $(f = 0 = g)$, and the system has an equilibrium at the origin $(H'_x(t,0,0) = 0 = H'_p(t,0,0))$. To make the results more readily available for use in the framework of classical mechanics, we have set up this study in the so-called classical case, where the Hamiltonian $H(t,x,p)$ is $p^2/2 + V(t,x)$, kinetic energy + potential.

We investigate the existence of periodic solutions other than the trivial one (rest at equilibrium). When V really depends on time, and is T-periodic in t, this is the study of so-called (nonlinear) parametric oscillations, with period an integer multiple of T. When V does not depend on time, this is the study of (anharmonic) free oscillations. The results are best illustrated by example (3.34), which is a kind of nonlinear n-dimensional Duffing's equation, with a time-varying parameter.

Previous results obtained through methods bearing some relation to the present one are described in [4] [6] [7] [8], and other techniques are applied, in the periodic case, by Rabinowitz [10] [11], Weinstein [13], Amann and Zehnder [1], among others (we refer to [11] for more detailed references). (For the most part, these deal with free (unforced) oscillations).

§2. Hamiltonian boundary-value problems.

We deal in this section with the following Hamiltonian system with forcing terms:

$$(2.1)\qquad (-\dot{p}(t),\ \dot{x}(t)) \in \partial H(t,\ x(t),\ p(t)) + (f(t),\ g(t)) \quad \text{a.e.}\quad ,$$

where $H: [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a given Hamiltonian and $f,\ g: [0,\infty) \to \mathbb{R}^n$ are given functions. We shall suppose that f and g belong to $L^2(0,a)$ for every finite positive a, and concerning the Hamiltonian $H(t,x,p)$ we make the following assumptions: $H(\cdot,x,p)$ is measurable for each (x,p), $H(t,\cdot,\cdot)$ is convex for each t, and there exist positive constarts $c,\ k,\ c',\ k',\ \alpha$ such that H satisfies, for all (t,x,p), the following:

$$(2.2)\qquad \frac{k'}{1+\alpha}\,|(x,p)|^{1+\alpha} - c' \le H(t,x,p) \le \frac{k}{2}\,|(x,p)|^2 + c \quad .$$

The ∂H in (2.1) refers of course to the subdifferential [12] of the convex function $(x,p) \to H(t,x,p)$, and as pointed out in §1, if it is further supposed that H is differentiable in (x,p) (which, however, we have no need to do), then (2.1) reduces to the familiar system of equations

$$\dot{x} = H_p'(t,x,p) + g(t)$$

$$-\dot{p} = H_x'(t,x,p) + f(t) \quad .$$

We denote $G(t,y,q)$ the conjugate of $H(t,\cdot,\cdot)$:

$$(2.3)\qquad G(t,y,q) = \sup\ \{y\cdot x + q\cdot p - H(t,x,p)\ :\ (x,p) \in \mathbb{R}^n \times \mathbb{R}^n\} \quad .$$

Our hypotheses imply that G is finite everywhere, measurable in t, and satisfies the following growth conditions, which are elementary consequences of combining (2.2) and (2.3) (see [6, Lemma 1]):

$$(2.4)\qquad G(t,y,q) \ge |(y,q)|^2/(2k) - c$$

$$(2.5)\qquad G(t,y,q) \le (k')^{-\beta}(1+\beta)^{-1}|(y,q)|^{1+\beta} + c' \quad ,$$

where $\beta = 1/\alpha$.

Let $F(t) = \int_0^t f(\tau)d\tau$, and observe that F belongs to $L^2(0,a)$ for any finite positive a. In the proofs of the theorems that follow, a certain functional I on $L^2 \times L^2$ intervenes repeatedly; we now define that functional and prove a useful existence theorem. Let φ and ψ be elements of $L^2((0,T); \mathbb{R}^n)$ for fixed $T > 0$, and define

$$(2.6)\quad I(\varphi,\psi) = \int_0^T \{\varphi(t) \cdot F(t) - [\varphi(t) + g(t)] \cdot \int_0^t \psi(\tau)d\tau + G(t, -\psi(t), \varphi(t))\}dt$$

Note that the integral is always well-defined, albeit perhaps as $+\infty$.

Theorem 2.1 If $T < \sqrt{2}/k$, then the functional I attains a minimum on any nonempty weakly closed subset S of $L^2 \times L^2$ upon which it is not identically $+\infty$.

Proof: (all norms are L^2) By Hölder's inequality we have

$$(2.7)\quad \|\int_0^t \psi(\tau)d\tau\| \leq T\|\psi\|/\sqrt{2} \quad ,$$

and if this is invoked along with (2.4) we deduce immediately

$$I(\varphi,\psi) \geq -\|\varphi\|\,[\|F\| + T\|\psi\|/\sqrt{2}] - T\|g\|\,\|\psi\|/\sqrt{2} + (\|\varphi\|^2 + \|\psi\|^2)/(2k) - c$$

Calling upon the inequality $2ab \leq a^2 + b^2$ produces

$$I(\dot{y},\dot{q}) \geq \{1/k - T/\sqrt{2}\}\ (\|\varphi\|^2 + \|\psi\|^2)/2 - \|F\|\,\|\varphi\| - T\|g\|\,\|\psi\|/\sqrt{2} - c$$

Since the coefficient of the quadratic term is positive, it follows that I is coercive; i.e. that for some $\varepsilon > 0$ and some constant $\tilde{c}$, I satisfies

$$(2.8)\quad I(\varphi,\psi) \geq \varepsilon(\|\varphi\|^2 + \|\psi\|^2) - \tilde{c}$$

Now let (φ_n,ψ_n) be a minimizing sequence for I over S. Then by (2.8), the subsequence is bounded in norm, so that we may select (without relabeling) a subsequence converging weakly to a limit (φ_0,ψ_0) in S. We deduce from dominated convergence that $\int_0^t \psi_n$ converges uniformly to $\int_0^t \psi_0$, whence follows the fact that

$$\int_0^T \{\varphi_n \cdot F - [\varphi_n + g] \cdot \int_0^t \psi_n\}dt$$

converges to the corresponding expression for (φ_0,ψ_0). The other component of I, the map $(\varphi,\psi) \to \int_0^T G(t,-\dot\psi,\dot\varphi)dt$, defines a convex functional known to be weakly lower semicontinuous (see [9] for details), and so we gather the threads to obtain

$$I(\varphi_0,\psi_0) \le \liminf I(\varphi_n,\psi_n) = \inf_S I \quad .$$

Q.E.D.

The preceding theorem is the key to the proofs of the existence theorems that follow. The solutions (x,p) which are asserted to exist are furthermore characterized as solutions to a specific minimization problem in the calculus of variations.

Theorem 2.2 Let $T < \sqrt{2}/k$, and let Δ_1, Δ_2 be any pair of points in $\mathbb{R}^n$. Then there exists on $[0,T]$ a solution (x,p) of the system (2.1) satisfying

$$x(T) = x(0) + \Delta_1 \quad , \quad p(T) = p(0) + \Delta_2 \quad .$$

Remark. If it so happens that, uniformly in t,

$$H(t,x,p)/|(x,p)|^2 \to 0 \quad \text{as} \quad |(x,p)| \to \infty \quad , \tag{2.9}$$

then the upper bound on H in (2.2) is satisfied (for appropriate c) by arbitrarily small k, so that the conclusions of the theorem hold for all T.

When Δ_2 has a certain specific value, our approach yields a slightly better upper bound for T (of course, the symmetry between x and p yields an analogous result for Δ_1):

Theorem 2.3 Let $T < 2\pi/k$, and let Δ_1 be any point in $\mathbb{R}^n$. Then there exists on $[0,T]$ a solution (x,p) of the system (2.1) satisfying

$$x(T) = x(0) + \Delta_1 \quad , \quad p(T) = p(0) - \int_0^T f(\tau)d\tau \tag{2.10}$$

Remark. Of course, the greatest interest of theorems such as the above has traditionally been in the study of periodic solutions. We obtain periodic solutions by postulating further that f, g, and $H(\cdot,x,p)$ be T-periodic, and by taking $\Delta_1 = \Delta_2 = 0$ in Theorem 2.2, or by supposing additionally that $\int_0^T f(\tau)d\tau = 0$ in Theorem 2.3.

The following result gives very precise conditions on x at the expense of saying nothing about the boundary values of p; it subsumes an earlier result of Aubin-Ekeland [2, Proposition 2].

Theorem 2.4 Let $T < \sqrt{2}/k$, and let x_0, x_1 be any pair of points in $\mathbb{R}^n$. Then there exists a solution (x,p) of system (2.1) satisfying

$$x(0) = x_0 \quad , \quad x(T) = x_1 \quad . \tag{2.11}$$

Proof of Theorem 2.2

Consider the problem of minimizing the functional I given by (2.6) subject to the conditions

$$\int_0^T \varphi(\tau)d\tau = \Delta_1 - \int_0^T g(\tau)d\tau \quad , \quad \int_0^T \psi(\tau)d\tau = \Delta_2 + \int_0^T f(\tau)d\tau \tag{2.12}$$

Note that (2.12) defines a weakly closed subset S of $L^2 \times L^2$, and that I is finite somewhere on S (for example, for the constant functions that belong to S). Thus by Theorem 2.1 the above minimization problem admits a solution $(\bar{\varphi},\bar{\psi})$. We now restate this fact so as to make it clear that we are dealing with a variational problem; to this end, define

$$\bar{y}(t) = \int_0^t \bar{\varphi}(\tau)d\tau \quad , \quad \bar{q}(t) = \int_0^t \bar{\psi}(\tau)d\tau \quad ,$$

and define the variational integrand L by:

$$L(t,y,q,\dot{y},\dot{q}) = \dot{y} \cdot F - [\dot{y} + g]q + G(t,-\dot{q},\dot{y})$$

Observe then that $(\bar{y},\bar{q})$ is a solution to the problem of minimizing

$$\int_0^T L(t,y,q,\dot{y},\dot{q})dt \tag{2.13}$$

over the absolutely continuous arcs (y,q) having derivatives in L^2 satisfying

$$y(T) - y(0) = \Delta_1 - \int_0^T g(\tau)d\tau \quad , \quad q(0) = 0 \quad , \quad q(T) = \Delta_2 + \int_0^T f(\tau)d\tau \tag{2.14}$$

Now if G were continuously differentiable (which is equivalent to H being strictly convex), it would follow that $(\bar{y},\bar{q})$ is an extremal for L, in the usual sense that it would satisfy the Euler-Lagrange equation corresponding to L. In our present setting, L is not differentiable (nor is it convex), however it is locally Lipschitz, and the results of [3] are available. These state that $(\bar{y},\bar{q})$ is an extremal for L in an extended sense: there exists an absolutely continuous function (r,s) such that

$$(2.15) \qquad (\dot{r},\dot{s},r,s) \in \partial L(t,\bar{y},\bar{q},\dot{\bar{y}},\dot{\bar{q}}) \qquad \text{a.e.}$$

where ∂L refers to the generalized gradient (see [3]) of L (in all variables jointly except t). (If L were C^1, then ∂L would reduce to the usual gradient; the reader is invited to show that in this case (2.15) is just the usual Euler-Lagrange equation). In view of how L is defined, (2.15) is equivalent to

$$(2.16) \qquad \dot{r} = 0 \quad , \quad \dot{s} = -\dot{\bar{y}} - g \qquad \text{a.e.}$$

$$(2.17) \qquad (-s,\ r - F + \bar{q}) \in \partial G(t,-\dot{\bar{q}},\dot{\bar{y}}) \qquad \text{a.e.}$$

It follows that for certain constants r_0 and σ_0, we have

$$(2.18) \qquad (\bar{y} + \int_0^t g(\tau)d\tau + \sigma_0,\ \bar{q} - F + r_0) \in \partial G(t,-\dot{\bar{q}},\dot{\bar{y}}) \qquad \text{a.e.}$$

By the inversion formula for conjugate subdifferentials, (2.18) is equivalent to

$$(2.19) \qquad (-\dot{\bar{q}},\dot{\bar{y}}) \in \partial H(t,\ \bar{y} + \int_0^t g + \sigma_0,\ \bar{q} - F + r_0) \qquad \text{a.e.}$$

Let us set

$$(2.20) \qquad x(t) = \bar{y}(t) + \int_0^t g(\tau)d\tau + \sigma_0$$

$$(2.21) \qquad p(t) = \bar{q}(t) - F(t) + r_0$$

Then (2.19) is just the system (2.1):

$$(-\dot{p} - f,\ \dot{x} - g) \in \partial H(t,x,p) \qquad \text{a.e.}$$

The fact that extremals of (2.13) are linked to (2.1) this way will be used repeatedly (with various types of boundary conditions). We summarize the result as follows:

Lemma 2.1 If $(\bar{y},\bar{q})$ is a solution to the problem of minimizing (2.13) subject to some set of boundary conditions on $(\bar{y},\bar{q})$ at 0 and T, then there exist c_0 and r_0 such that the translates x and p of $\bar{y}$ and $\bar{q}$ defined by (2.20) and (2.21) satisfy (2.1).

Remark. A functional analytic proof of essentially this lemma, independent of the results in [3], and based upon an argument in [5], is given in [6].

Returning now to the proof of Theorem 2.2, we note that our present boundary conditions (2.14) imply

$$x(T) - x(0) = \Delta_1 \quad , \quad p(T) - p(0) = \Delta_2$$

Q.E.D.

Proof of Theorem 2.3

Lemma 2.2 If $T < 2\pi/k$, then there exist positive constants ε_1 and c_1 such that, *for all* φ, ψ *in* L^2, *when* $\int_0^T \psi(\tau)d\tau = 0$, *we have*

$$I(\varphi,\psi) \geq \varepsilon_1(\|\varphi\|^2 + \|\psi\|^2) - c_1 \tag{2.22}$$

Proof: When $\int_0^T \psi(\tau)d\tau = 0$, we have the estimate

$$\left\|\int_0^t \psi(\tau)d\tau\right\| \leq T\|\psi\|/(2\pi)$$

(This may be seen, for example, by comparing the Fourier expansion $\sum_{j=0}^{\infty} q_j e^{2\pi ijt/T}$ of the T-periodic function $q(t) = \int_0^t \psi(\tau)d\tau$ with that of $\dot{q} = \psi$, upon noting that $q_0 = 0$). When this sharper estimate replaces (2.7), the same proof that led to (2.8) yields (2.22).

It then follows as before that the functional I admits a minimum over the class defined by (2.12), with $\Delta_2 = \int_0^T f(\tau)d\tau$. The proof from this point on is identical to that of Theorem 2.2.

Q.E.D.

Proof of Theorem 2.4

We consider the problem of minimizing the functional

$$x_1 \cdot \int_0^T \psi(\tau)d\tau + I(\varphi,\psi) \tag{2.23}$$

where I is given by (2.6) as before. Note that if $\tilde{G}$ is defined by

$$\tilde{G}(t,-\psi,\varphi) = G(t,-\psi,\varphi) + x_1 \cdot \psi \tag{2.24}$$

then *minimizing* (2.23) *is equivalent to minimizing the functional* $\tilde{I}$ *defined exactly* as in (2.6), but with $\tilde{G}$ replacing G. If we pick any $\tilde{k} > k$, then $\tilde{G}$ will satisfy a condition (2.4) for c replaced by a larger $\tilde{c}$, since it differs from G by only a linear term. Similarly, $\tilde{G}$ will satisfy a condition of the form (2.5). Now we may *also* pick $\tilde{k}$ so that

$$T < \sqrt{2}/\tilde{k} \quad ,$$

so that Theorem 2.1 will now apply to enable us to conclude that there is a solution $(\bar{\varphi},\bar{\psi})$ to the problem of minimizing (2.23) subject to the condition $\int_0^T \varphi(\tau)d\tau = x_1 - x_0 - \int_0^T g(\tau)d\tau$. Equivalently, in a more classical formulation, the arc

$$(\bar{y},\bar{q}) = \left(\int_0^t \bar{\varphi}, \int_0^t \bar{\psi}\right)$$

minimizes

$$x_1 \cdot q(t) + \int_0^T L(t,y,q,\dot{y},\dot{q})dt \tag{2.5}$$

subject to the boundary constraints

$$(2.26) \qquad q(0) = 0 \quad , \quad y(T) - y(0) = x_1 - x_0 - \int_0^T g(\tau)d\tau \quad .$$

It follows precisely as in the proof of Theorem 2.2 that $(\bar{y},\bar{q})$ is an extremal for L, which implies (Lemma 2.1) that for certain constants σ_0, r_0, the functions x and p defined by (2.20) (2.21) satisfy (2.1). The fact that $q(T)$ is unspecified and appears as it does in the Bolza functional (2.25) leads as well to the conclusion that $s(T) = -x_1$, where s is the "adjoint variable" that appeared in (2.15). (This is nothing more than the "transversality condition" of [3, Theorem 1]). We have already seen in (2.16) (2.17) that

$$-s(t) = \bar{y}(t) + \int_0^t g(\tau)d\tau + \sigma_0 = x(t) \quad ,$$

so that (in light of (2.26)) we deduce

$$x(T) = x_1 \quad , \quad x(0) = x(T) - \int_0^T \dot{x}(\tau)d\tau = x_0 \quad .$$

Q.E.D.

§3. The classical case: free vibrations.

We now investigate the case when there is no forcing term on the right-hand side. The Hamiltonian now is:

$$H(t,x,p) = V(t,x) + \frac{1}{2} p^2 \tag{3.1}$$

and Hamilton's equations are:

$$\dot{x} = p \quad , \quad \dot{p} = -\partial V(t,x) \quad , \tag{3.2}$$

which, in the particular case when V is differentiable (C^2) and does not depend on time, boils down to the second-order system (Newton's equations):

$$\ddot{x} = -V'(x) \tag{3.3}$$

The existence results in the preceding sections certainly apply to this case. However, a new problem arises in this context: non-triviality. If $x_0 \in \mathbb{R}^n$ is an equilibrium, i.e. if $\partial_x V(t,x_0) \ni 0$ for all t (since $V(t,\cdot)$ is convex, x_0 then is a global minimum) then the constant solution $x(t) = x_0$, $p(t) = 0$ is T-periodic for all $T > 0$. This we call a trivial solution: the system rests at x_0. The following theorem asserts the existence of non-trivial solutions, i.e. free vibrations of the system.

Theorem 3.1 Assume the potential $V(t,x)$ is measurable with respect to $t \in [0,T]$, strictly convex with respect to $x \in \mathbb{R}^n$. Assume that the origin is an equilibrium:

$$\forall (t,x) \quad , \quad V(t,x) \geq V(t,0) = 0 \tag{3.4}$$

and that the following estimates hold:

$$\forall (t,x) \qquad \frac{k'}{1+\alpha} |x|^{1+\alpha} - c' \leq V(t,x) \leq \frac{k}{2} x^2 + c \tag{3.5}$$

$$\forall t \qquad |x| \leq \eta \rightarrow V(t,x) \geq \frac{K}{2} x^2 \tag{3.6}$$

where k', α, c', k, c, η, K are strictly positive constants.

If $T \in (\frac{2\pi}{K}, \frac{2\pi}{k})$, then there is at least one non-trivial returning solution $(x(t), p(t))$ of Hamilton's equations (3.2):

$$(x(0), p(0)) = (x(T), p(T)) \tag{3.7}$$

The proof, of course, will involve the dual action principle, which itself involves the Legendre transform $U(t,\cdot)$ of the convex function $V(t,\cdot)$ given by Fenchel's formula:

(3.8) $$U(t,y) = \operatorname*{Sup}_{x} \{xy - V(t,x)\} .$$

We claim that there exists some $\varepsilon > 0$ such that:

(3.9) $$\forall t, \quad |y| \le \varepsilon \Rightarrow U(t,y) \le \frac{2}{K} y^2$$

To prove this, we simply write that for all t:

(3.10) $$\mathrm{Sup}\,\{xy - V(t,x) \mid |x| \le \eta\} \le \mathrm{Sup}\,\{xy - \frac{K}{2} x^2 \mid |x| \le \eta\}$$

because of relation (3.6), so that:

(3.11) $$\mathrm{Sup}\,\{xy - V(t,x) \mid |x| \le \eta\} \le 2y^2/K .$$

The function $x \mapsto xy - V(t,x)$ attains its (unconstrained) maximum for $x \in \partial_y U(t,y)$, and its value then is $U(t,y)$. In other words, the left-hand side of inequality (3.11) coincides with $U(t,y)$, as long as $\partial_y U(t,y)$ intersects the ball of radius η around the origin.

But $\partial U(t,0)$ is known to be $\{0\}$, because it is just the set of points where $V(t,\cdot)$ attains its minimum, which is $\{0\}$ because of inequality (3.6). This same estimate enables us to state that:

(3.12) $$V(t,x) \ge K\eta\,|x|/2 \quad \text{for} \quad |x| \ge \eta$$

and hence:

(3.13) $$V(t,x) \ge \varphi(|x|) \quad \text{all} \quad (t,x)$$

with $\varphi(t) = Kt^2/2$ for $|t| \le \eta/2$ and $\varphi(t) = K\eta(t - \eta/2)/2$ for $|t| \ge \eta/2$. It follows that:

(3.14) $$U(t,x) \le \varphi^*(|x|) \quad \text{all} \quad (t,x) \quad ,$$

where φ^* is the Legendre transform of φ. It is finite for $|t| \le K\eta/2$, so that, for all t, the function U is continuous on the ball of radius $K\eta/2$ around the origin. It follows that, on this ball, the multi-valued mapping $y \mapsto \partial_y U(t,y)$ is well-defined, with non-empty, convex, compact values, and is upper semi-continuous. Since $\partial_y U(t,0) = \{0\}$, this means that there is some $\varepsilon < K\eta/2$ such that,

$$|y| < \varepsilon \Rightarrow \exists x \in \partial_y U(t,y) : |x| < \eta \tag{3.15}$$

For $|y| < \varepsilon$, the left-hand side of (3.11) is just $U(t,y)$; hence formula (3.9).

We now prove the theorem

<u>Proof</u>: Consider, as in §2, the path $(y(t), q(t))$ which minimizes the dual action integral:

$$J(y,q) = \int_0^T \{-\dot{y}(t)q(t) + \frac{1}{2}\dot{y}(t)^2 + U(t, -\dot{q}(t))\}dt \tag{3.16}$$

under the constraints:

$$\int_0^T \dot{y}(t)dt = 0 \quad , \quad \int_0^T \dot{q}(t)dt = 0 \tag{3.17}$$

From Theorem 2.1 (modified by lemma 2.2), with $f = 0 = g$, there is a minimizing $(\dot{y},\dot{q})$ in $L^2 \times L^2$, which corresponds to a solution (x,p) of Hamilton's equations:

$$x(t) = U_y'(t, -\dot{q}(t)) = y(t) + y_0 \tag{3.18}$$

$$p(t) = \dot{y}(t) = q(t) + q_0 \tag{3.19}$$

Clearly, $x(T) = x(0)$ and $p(T) = p(0)$. We now show non-triviality. If $(x(t), p(t))$ were identically $(0,0)$ then so would $(-\dot{q}(t), \dot{y}(t))$ because of (3.18) and (3.19) (relation (3.6) implies that 0 is the only critical point of $U(t,\cdot)$). This in turn would imply that $J(y,q)$, the minimum value of J, is zero.

We will now find a feasible path (y_0,q_0) such that $J(y_0,q_0) < 0$. Start off by writing J slightly differently:

$$J(y,q) = \int_0^T \{\frac{1}{2}(\dot{y}(t) - q(t))^2 - \frac{1}{2}q(t)^2 + U(t, -\dot{q}(t))\}dt$$

Using estimate (3.9)

$$J(y,q) \leq \frac{1}{2}\int_0^T \{\dot{y}(t) - q(t)\}^2 dt + \frac{1}{2}\int_0^T \{\frac{1}{K}\dot{q}(t)^2 - q(t)^2\}dt$$

Taking $q_0(t) = a \cos (\frac{2\pi}{T} t)$ and $y_0(t) = \frac{aT}{2r} \sin (\frac{2\pi}{T} t)$, with $0 < |a| \varepsilon$ we get:

$$J(y_0,q_0) \leq \frac{1}{2} (\frac{1}{K} \frac{4\pi^2}{T^2} - 1) \frac{a^2}{2} ,$$

which is strictly negative if $T < 2\pi/K$. Hence the result.

If $V(\cdot,x)$ is T-periodic for each fixed x in $\mathbb{R}^n$, then the relations (3.7) obviously imply the existence of a T-periodic solution $(x(t), p(t))$. Such a solution obviously is kT-periodic for all $k \in \mathbb{N}$. It also could be k^{-1}T-periodic for some $k \in \mathbb{N}$, except if T is the minimal period. Interesting questions arise in that connection: how many T-periodic solutions are there? Is there a way of finding one with minimal period T?

The following results provide partial answers to these questions. From now on, we identify T-periodic functions defined on $\mathbb{R}$ with their restrictions on $[0,T]$.

<u>Theorem 3.2</u> Let $V(t,x)$ be T-periodic in t for all $x \in \mathbb{R}^n$, and satisfy all the assumptions of theorem 3.1. Let $(x(t), p(t))$ be some T-periodic solution found by minimizing the dual action integral on $[0,T]$, and $(y(t), q(t))$ the corresponding minimizer. Set:

(3.20) $$\forall y \in \mathbb{R}^n , \Delta(y) = \operatorname{Sup} \{|U(s,y) - U(s',y)| \mid (s,s') \in \mathbb{R}^2\}$$

(3.21) $$A = 1 + (\int_0^T \Delta(-\dot{q}(t))dt) (\int_0^T \dot{y}(t)q(t)dt)^{-1} .$$

(We show below that the quantity whose reciprocal is taken is strictly positive.)

Then, for no integer $k > A$ can the solution $(x(t), p(t))$ be k^{-1}T-periodic.

<u>Proof</u>: Assume $(x(t), p(t))$ is k^{-1}T-periodic. Then so is the function $(y(t), q(t))$ because of relations (3.18) and (3.19). We define a new, T-periodic, function $(y_k(t), q_k(t))$ by:

$$(y_k(t), q_k(t)) = (ky(tk^{-1}), kq(tk^{-1}))$$

We now compute $J(y_k, q_k)$ and get:

$$J(y_k,q_k) = \int_0^T \{-\dot{y}_k(t)q_k(t) + \frac{1}{2}\dot{y}_k(t)^2 + U(t, -\dot{q}_k(t))\}dt$$

$$= \int_0^T \{-k\dot{y}(tk^{-1})q(tk^{-1}) + \frac{1}{2}\dot{y}(tk^{-1})^2 + U(t, -\dot{q}(tk^{-1}))\}dt$$

$$= \int_0^{Tk^{-1}} \{-k\dot{y}(s)q(s) + \frac{1}{2}\dot{y}(s)^2 + U(ks, -\dot{q}(s))\}k\,ds$$

The integrand is $k^{-1}T$-periodic, so that its integral over $[0, k^{-1}T]$ is k^{-1} times its integral over $[0,T]$. This yields:

$$J(y_k,q_k) = \int_0^T \{-k\dot{y}(s)q(s) + \frac{1}{2}\dot{y}(s)^2 + U(ks, -\dot{q}(s))\}ds$$

$$= J(y,q) - (k-1)\int_0^T \dot{y}(s)q(s)ds + \int_0^T \{U(ks, -\dot{q}(s)) - U(s, -\dot{q}(s))\}ds$$

$$J(y_k,q_k) \le J(y,q) - (k-1)\int_0^T \dot{y}(s)q(s)ds + \int_0^T \Delta(-\dot{q}(s))ds$$

But $J(y_k,q_k)$ has to be greater or equal to $J(y,q)$, since (y,q) is a minimizer. The result follows immediately.

In other words, formula (3.21) yields an a priori estimate for the minimal period of the solution $(x(t), p(t))$ found in theorem 3.1. For this estimate to be meaningful, the right-hand side of (3.21) has to be finite: we shall see that this is the case in general.

Notice first that the integral $\int_0^T \dot{y}q\,dt$ is strictly positive. Indeed, we have, by formula (3.16):

$$-\int_0^T \dot{y}q\,dt = \operatorname{Min} J - \int_0^T \{\frac{1}{2}\dot{y}(t)^2 + U(t, -\dot{q}(t))\}dt$$

The first term on the right-hand side is negative as seen in the proof of theorem 3.1, and the second is too, because of inequality (3.4):

$$U(t,y) = \sup_x \{xy - V(t,x)\} \geq 0 \cdot y - V(t,0) = 0$$

We now show that, under slightly stronger assumptions then those of theorem 3.1, the estimate A is finite. Indeed, inequalities (3.5) yield:

$$\frac{1}{2k} y^2 - c \leq U(t,y) \leq \frac{1}{(1+\beta)k'} |y|^{1+\beta} + c'$$

with $\beta = 1/\alpha$. This gives us in turn:

$$\Delta(y) \leq (c' + c) + (k' + \beta k')^{-1} |y|^{1+\beta} + (2k)^{-1} |y|^2$$

$$\int_0^T \Delta(-\dot{q}(t))dt \leq (c' + c)T + (k' + \beta k')^{-1} \|\dot{q}\|_{1+\beta}^{1+\beta} + (2k)^{-1} \|\dot{q}\|_2^2$$

Going back to formula (3.21), we see that a necessary condition for A to be finite is that $\dot{q} \in L^{1+\beta}(0,T;\mathbb{R}^n)$, with $\beta = \alpha^{-1}$. But $\dot{q}$ is defined by formula (3.18); using the Legendre reciprocity formula:

$$\dot{q}(t) \in -\partial_x V(t, x(t))$$

where $x(t)$ is known to be continuous, and hence bounded. This will in general be enough to ensure that $\dot{q}$ belongs to $L^{1+\beta}$. For instance, if the potential $V(t,x)$ is continuous with respect to both variables, then the set:

$$\cup\, \partial_x V(t,x) \quad \text{over all} \quad (t,x) \in [0,T] \times B$$

is bounded whenever the set $B \subset \mathbb{R}^n$ is bounded, so that $\dot{q} \in L^\infty \subset L^{1+\beta}$, as desired.

We state a few simple consequences:

Corollary 3.1 Assume the potential $V(t,x)$ is continuous with respect to the variables (t,x), strictly convex in $x \in \mathbb{R}^n$ and T-periodic in $t \in \mathbb{R}$. Assume that the origin is an equilibrium:

(3.22) $$\forall (t,x) \quad V(t,x) \geq V(0,0) = 0$$

and that the following estimates hold:

(3.23) $$\forall (t,x) \quad \frac{k'}{1+\alpha} |x|^{1+\alpha} - c' \leq V(t,x) \leq C\, |x|^{2-\gamma} + c$$

(3.24) $$\forall t, \quad |x| \leq \eta \Rightarrow V(t,x) \geq \frac{K}{2} x^2$$

where k', a, c', C, γ, c, η, K are strictly positive constants, with $T > 2r K^{-1}$.

Then there is an infinite sequence $k_0 = 1, \ldots, k_n, \ldots$ of integers, and for each k_n a $k_n T$-periodic solution $(x_n(t), p_n(t))$ of Hamilton's equations (2) such that

(3.25) $$m \neq n \Rightarrow (x_n(t), p_n(t)) \not\equiv (x_m(t), p_m(t))$$

Proof: Because of inequality (3.23), assumption (3.5) of theorem 3.1 is seen to hold for all $k > 0$ (by adjusting c if necessary). It follows that the process of minimizing the dual action integral can be applied to each of the periods kT, $k \in \mathbb{N}$, starting with $k = 1$, which yields some T-periodic solution $(x_0(t), p_0(t))$. By theorem 3.2, there is some integer k_1 such that the $k_1 T$-periodic solution $(x_1(t), p_1(t))$ obtained in this way is not T-periodic any more.

Now start the process again with the new period $T' = k_1 T$, and pick an integer k_2' and a solution $(x_2(t), p_2(t))$ which is $k_2' T'$-periodic but not T'-periodic. Set $k_2 = k_2' k_1$. Start again with $T'' = k_2' T' = k_2 T$. The solutions $(x_n(t), p_n(t))$ defined by that process must have different (increasing) minimal periods.

Corollary 3.2 Assume the potential $V(x)$ does not depend on time t, and is a strictly convex function of $x \in \mathbb{R}^n$. Assume the origin is an equilibrium:

(3.26) $$\forall x, \quad V(x) \geq V(0) = 0$$

and that the following estimates hold, with $k \leq K$:

(3.27) $$\forall x, \quad V(x) \leq \frac{k}{2} x^2 + c$$

(3.28) $$|x| \leq \eta \Rightarrow V(x) \geq \frac{K}{2} x^2$$

Then, for any $T \in (2\pi K^{-1}, 2\pi k^{-1})$, there is a periodic solution of Hamilton's equation:

(3.29) $$\dot{x} = p \quad , \quad \dot{p} \in -\partial V(x)$$

with minimal period T.

Proof: If the potential V satisfies another estimate:

$$\forall x, \quad \frac{k'}{1+\alpha}|x|^{1+\alpha} - c' \leq V(x) \quad , \tag{3.30}$$

for some strictly positive constants α, k', c', then all one has to do is to apply theorems 3.1 and 3.2. The potential V is T-periodic in t for all $T > 0$, and formula (3.20) defines $\Delta(y)$ to be identically zero, so that $A = 1$. The result follows.

If estimate (3.30) is not satisfied, one simply replaces V by another strictly convex potential V_R which does satisfy (3.30), in addition to (3.27), and which coincides with V on some large ball:

$$|x| \leq R \Rightarrow V_R(x) = V(x) \quad . \tag{3.31}$$

Now find, by minimizing the dual action integral, some T-periodic solution of the equations $\dot{x} = p$, $\dot{p} \in -\partial V_R(x)$, for which the Hamiltonian is a first integral:

$$\frac{1}{2}p(t)^2 + V_R(x(t)) = \text{constant} = h \quad . \tag{3.32}$$

This constant h can be estimated. We have, using (3.18), (3.19) and Legendre's formula:

$$\begin{aligned}\frac{1}{2}p(t)^2 + \frac{1}{2}\dot{y}(t)^2 + V_R(x(t)) + U_R(-\dot{q}(t)) &= \\ &= p(t)\dot{y}(t) - x(t)\dot{q}(t) \\ &= p(t)^2 - y(t)\dot{q}(t) - y_0\,\dot{q}(t)\end{aligned}$$

Integrating both sides and using (3.32):

$$\int_0^T h\,dt + \int_0^T \{-\dot{y}q + \dot{y}^2/2 + U_R(-\dot{q})\}dt = \int_0^T p^2\,dt$$

$$hT + J_R(y,q) = \int_0^T p^2\,dt$$

One then proceeds as in [6] to get the estimate:

$$h \leq \text{Max}\left[c, \frac{ckT}{2r - kT}\right] \tag{3.33}$$

which depends only on the constants in formula (3.27), and not on R.

The procedure is now obvious. Let $T > 2rk^{-1}$ be given. Formula (3.33) then gives an upper bound for h. Choose R so large that $V(x) \leq h$ implies $|x| \leq R$. Find $(x(t), p(t))$, a solution of Hamilton's equations for V_R, with minimal period T. By equation (3.32), $x(t)$ lies entirely within the ball $|x| \leq R$, so that $(x(t), p(t))$ solves Hamilton's equations for V.

We conclude this section by an example. The n-dimensional equation:

$$\ddot{x} + (1 + a \cos 2t)x + bx\left(\sum_{i=1}^{n} x_i^2\right)^{\beta} = 0 \tag{3.34}$$

has a non-trivial 2π-periodic solution provided

$$0 < |a| < 1 \quad , \quad -1 < \beta < 0 \quad , \quad b \neq 0 \tag{3.35}$$

This follows from theorem 3.1, with the potential

$$V(t,x) = (1 + a \cos 2t)x^2/2 + b\left(\sum_{i=1}^{n} x_i^2\right)^{\beta+1} \Big/ (2\beta + 2) \tag{3.36}$$

(note that K can be taken larger than any given number, because of the second term, and that $k = 1 + |a|$)

If $a = 0$, for any $T < 2\pi$, equation (3.34) has a solution with minimal period T.

If $b = 0$, equation (3.34) becomes linear, and does not have a 2π-periodic solution any more, because one encounters parametric resonance.

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We prove existence of solutions to various boundary-value problems for nonautonomous Hamiltonian systems with forcing terms:

$$-\dot{p}(t) = H'_x(t, x(t), p(t)) + f(t)$$

$$\dot{x}(t) = H'_p(t, x(t), p(t)) + g(t) \quad .$$

Among these problems is that of periodic solutions: $x(t + T) = x(t)$,

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ABSTRACT (continued)

$p(t + T) = p(t)$. A special study is made of the classical case in which $H(x,p)$ has the form $V(x) + |p|^2/2$, potential plus kinetic energy, where the existence of an infinite family of free harmonics is proven. The approach throughout is via a variational principle involving a new, dual action integral.